Systems Biology: Mathematics for Biologists

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Equilibrium types in 2D systems

Equilibrium types

1D systems:

- Two regions, left and right of an equilibrium.
- Arrows can point away or toward the equilibrium.
- So two equilibrium types possible: stable and unstable.
- At bifurcation point special case: stable and unstable sides.

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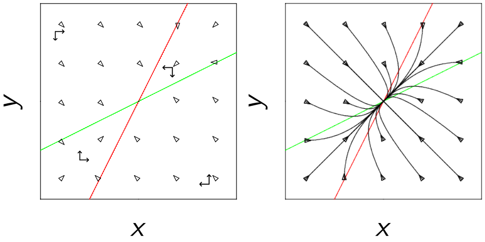
- Arrows can point away or toward the equilibrium.
- So two equilibrium types possible: stable and unstable.
- At bifurcation point special case: stable and unstable sides.

2D systems:

- Four regions around an equilibrium point.
- Arrows can point away or toward the equilibrium, *or both*!
- Six different equilibrium types possible, two of which are stable.

Equilibrium types: stable node

$$\begin{cases} \frac{dx}{dt} = -2x + y \\ \frac{dy}{dt} = x - 2y \end{cases}$$



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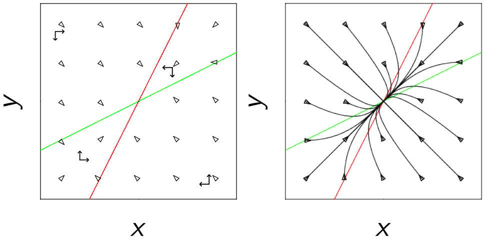

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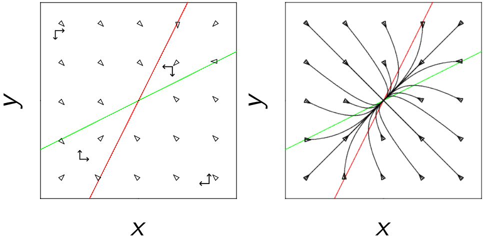

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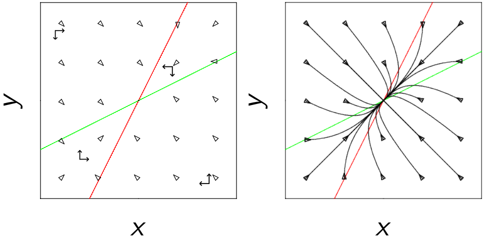

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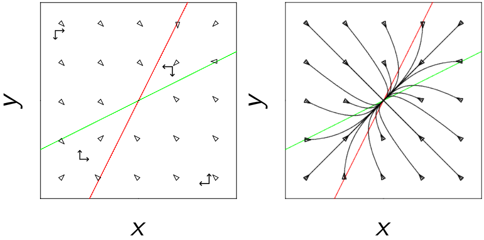

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Phase portrait gives same information as numerical solutions

Equilibrium types: stable node (2)

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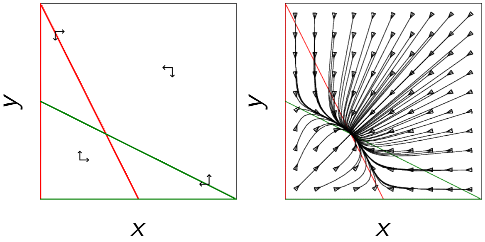

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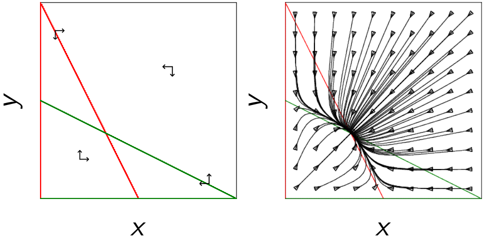

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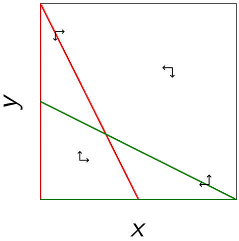

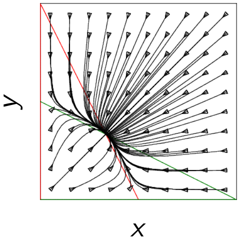

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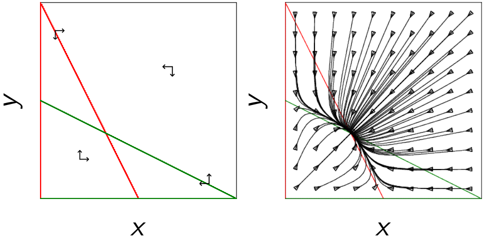

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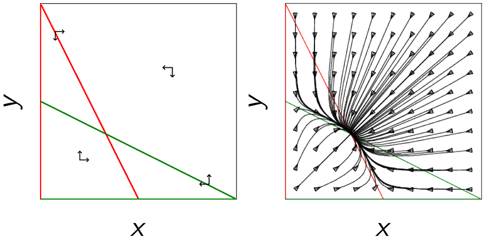

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Compare: different nullclines, similar vectorfield!

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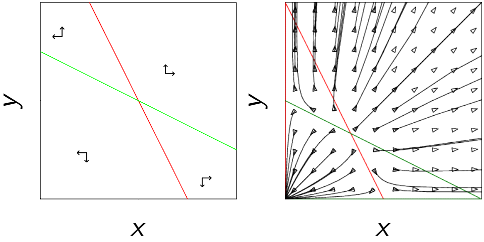

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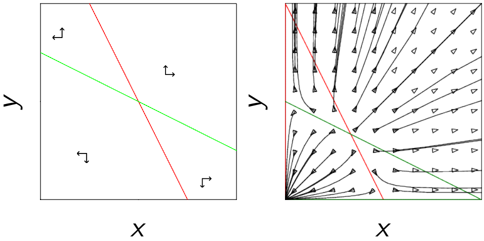

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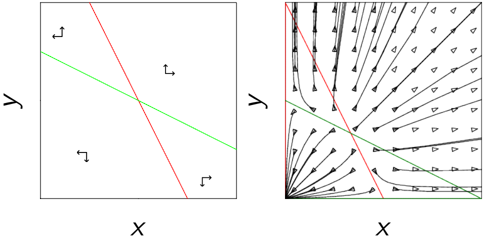

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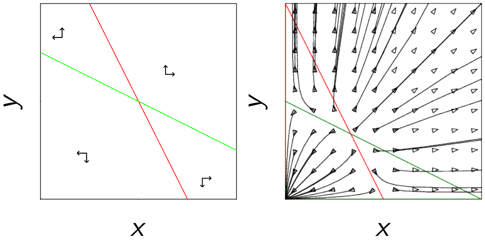

Vectorfield: all arrows away from equilibrium $\rightarrow$ **unstable node**

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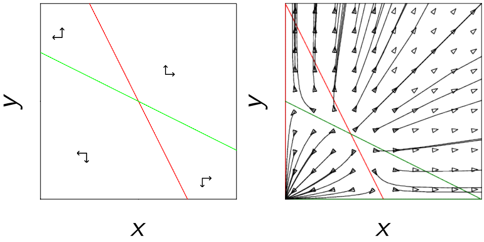
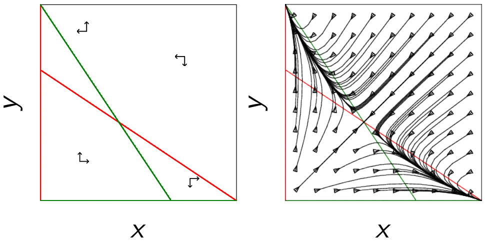


Vectorfield: all arrows away from equilibrium $\rightarrow$ **unstable node**

Compare: same nullclines, very different vectorfield

Equilibrium types: saddle point

$$\begin{cases} \frac{dx}{dt} = -x - 2y \\ \frac{dy}{dt} = -2x - y \end{cases}$$



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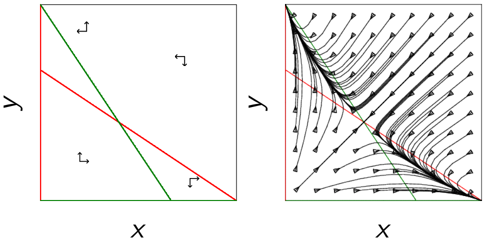

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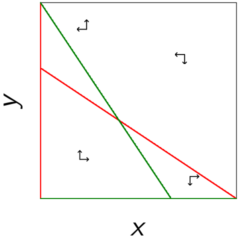

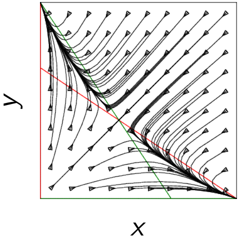

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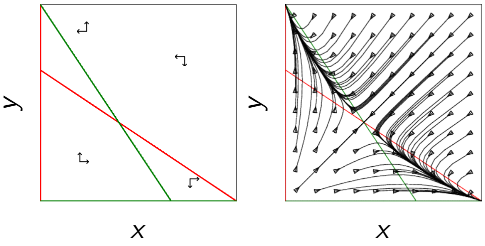

Vectorfield:

one vector-pair points towards, one points away from equilibrium: stable and unstable direction $\rightarrow$ **saddle point**

Equilibrium types: stable spiral

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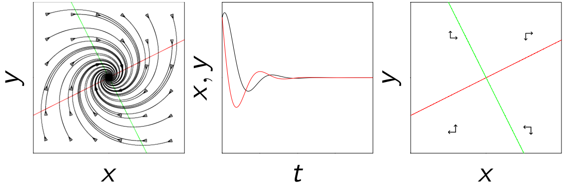

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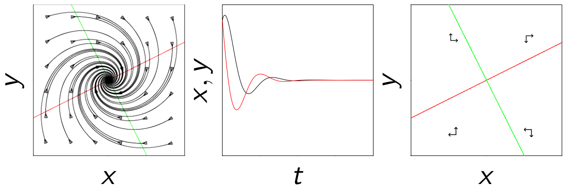

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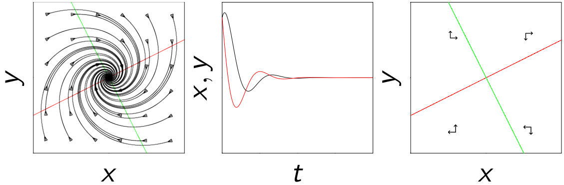

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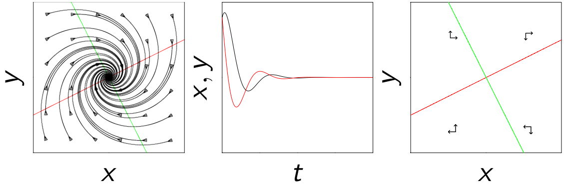

Inward spiraling motion towards equilibrium
Oscillations with decreasing amplitude

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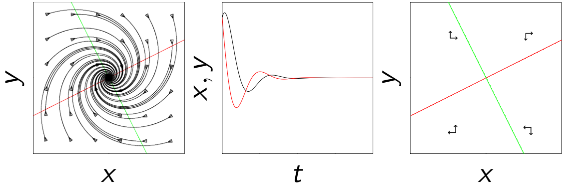

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Vectorfield: arrows only suggest rotation!

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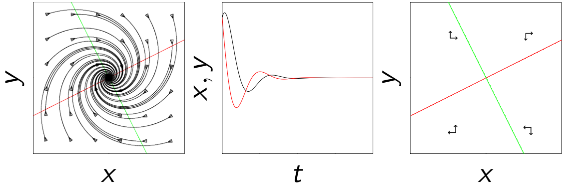

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Phase portrait gives less information than numerical solutions...

Equilibrium types: unstable spiral

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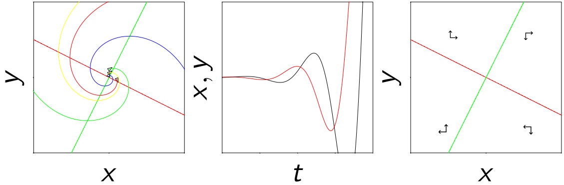

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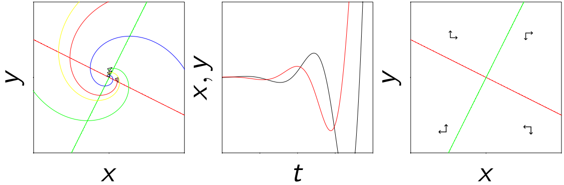

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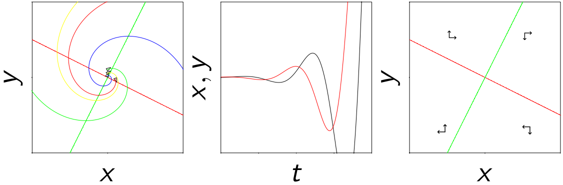

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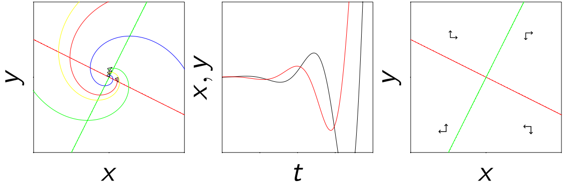

Outward spiraling motion away from equilibrium

Oscillations with increasing amplitude

Equilibrium types: unstable spiral

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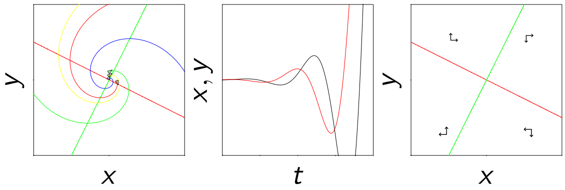

Outward spiraling motion away from equilibrium
Oscillations with increasing amplitude

Vectorfield: arrows again only suggest rotation!

Equilibrium types: center point

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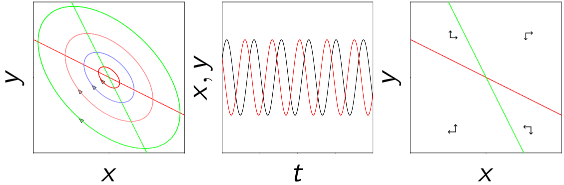

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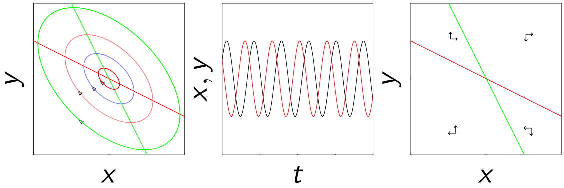

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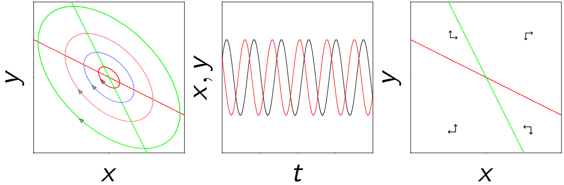

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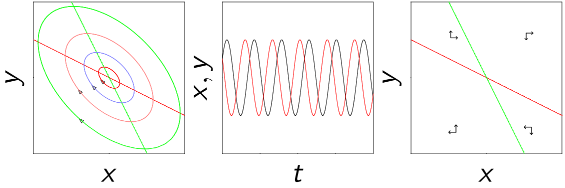

Rotation around equilibrium at constant distance
Oscillations amplitude determined by initial conditions

Equilibrium types: center point

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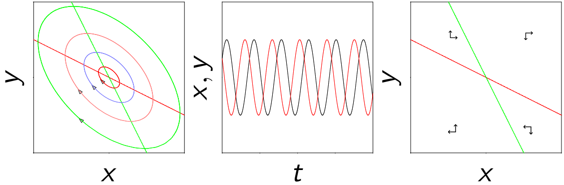
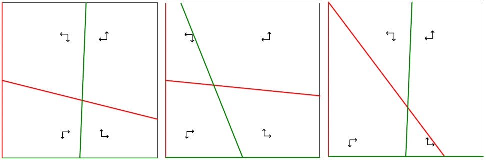


Rotation around equilibrium at constant distance
Oscillations amplitude determined by initial conditions

Vectorfield: arrows again only suggest rotation!

Vectorfield insufficient

Sometimes the vectorfield does not give enough information:



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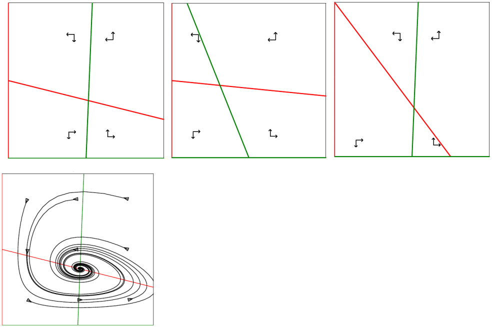

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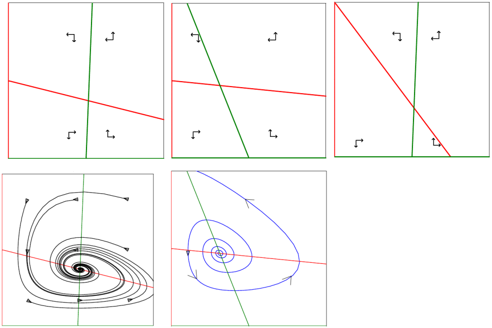

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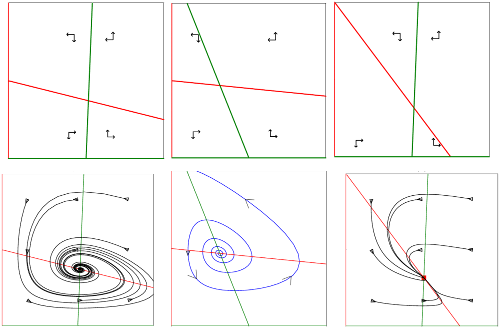

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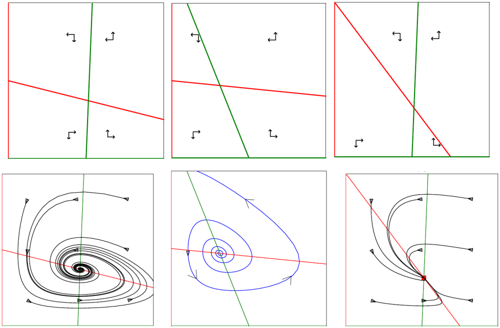

All vectorfields suggest rotation, but we may even have a node!

Self-feedback: when and how

First look at the entire vectorfield:
is it clearly a stable node, unstable node, saddle?

- **YES**: you are finished!
- **NO**: look at **self-feedback**

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- Feedback of x variable on itself
  - add a little x (small horizontal step from eq. to the right)
  - does x increase or decreases (horizontal vector to left or right)

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Self-feedback:

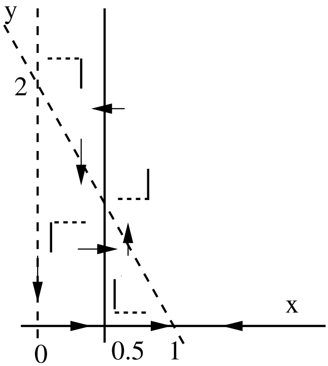
- Feedback of x variable on itself
  - add a little x (small horizontal step from eq. to the right)
  - does x increase or decreases (horizontal vector to left or right)
- Feedback of y variable on itself
  - add a little y (small vertical step from eq. upwards)
  - does y increase or decreases (vertical vector up or down)

Self-feedback: example

predator-prey system:

$$\begin{cases} \frac{dx}{dt} = 3x(1-x) - 1.5xy \\ \frac{dy}{dt} = 0.5xy - 0.25y \end{cases}$$

x null-clines:
x=0 and y=2-2x
y null-clines
y=0 and x=0.5



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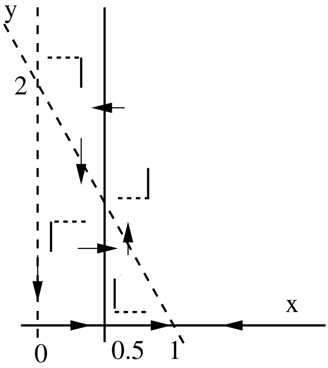

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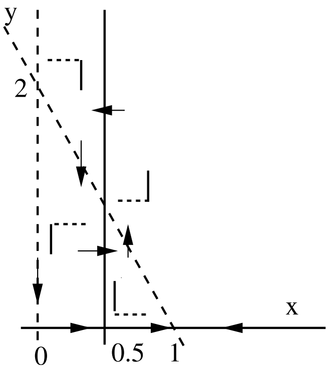

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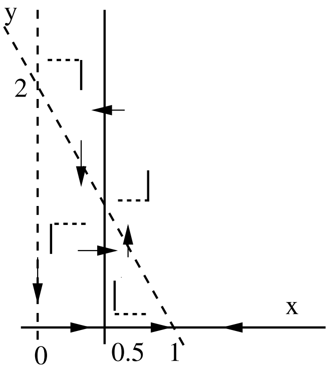

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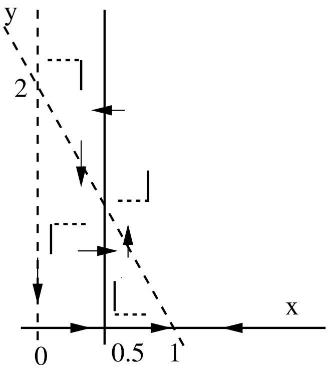

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stable equilibrium! (probably stable spiral)

Self-feedback: summary

Self-feedback and stability:

- **Stable**
  - x and y have negative feedback on themselves
  - x has negative and y has zero feedback on itself
  - x has zero and y has negative feedback on itself

Self-feedback: summary

Self-feedback and stability:

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Self-feedback: summary

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Net feedback negative $\rightarrow$ stable equilibrium
Net feedback positive $\rightarrow$ unstable equilibrium

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Note that from self-feedback we can not determine equilibrium type!

An overview of 2D equilibria

An equilibrium is only stable, if *all* directions converge on it. One or more diverging directions means that the equilibrium is unstable.

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  - saddle node: one stable and one unstable direction
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  - saddle node: one stable and one unstable direction
  - unstable spiral: rotation, net positive self-feedback
- A center point is **neutrally stable**:
  - rotation, net zero self-feedback
  - neither convergence nor divergence