Solving equations composed of variables and free parameters

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This is a short tutorial to fresh up some of your high school mathematics using examples with free parameters. Working with free parameters, instead of the numbers that you may be used to, mathematically makes little difference because the same rules apply, e.g.,

$$a+b+c=(a+b)+c=a+(b+c)\ , \quad a(b+c)=ab+bc\ , \quad \frac{a+b}{c+d}=\frac{a}{c+d}+\frac{b}{c+d}\ , \quad \frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}\ .$$

Just fill in integer numbers for the letters to check these expressions. We will use x, y and z as variables (that need to be solved), and a, b, c, d and e as parameters (constants). Let us agree here that all parameters are non-zero and positive (e.g., $a - b$ means the subtraction of two positive numbers).

This tutorial provides various exercises to improve your skills on each topic. Answers can be obtained by entering the equation of the exercise into the website of WolframAlpha. Equations can be solved by typing '`solve ax-b=0`' (or more explicitly '`solve ax-b=0 for x`'), and simplified by typing '`simplify a/b + c/d`'. One can add a `*` character, or a space, to make clear that `ax` means $a \times x$.

There are many tutorials on the web covering the same topics. The very best of them is the Khan Academy providing an excellent variety of videos on math. The only reason for duplicating a few of them here is our emphasis on parameter-free expressions. Most of the other tutorials use digits rather than letters, i.e., $3x + 2 = 0$ instead of $ax + b = 0$. Finally, for Dutch students we highly recommend the book 'Basisboek wiskunde' by Jan van de Craats and Rob Bosch (a pdf can be downloaded from `https://staff.fnwi.uva.nl/j.vandecraats/BasisboekWiskunde2HP.pdf`), which provides short and crisp-clear explanations, and lots of exercises. As always, the skills explained in this short tutorial can only be absorbed by making lots of exercises.

1 Linear equations

The procedure for solving equations of the form $f(x, y) = g(x, y)$, where $f()$ and $g()$ are arbitrary expressions in terms of some variables x and y, is to first simplify both sides by modifying them equally. There are two rules for modification: **(1)** one can always add a factor to (or subtract from) the left- and right-hand side simultaneously, and **(2)** one can always multiply (or divide) both sides of the equation with (by) the same factor. For instance, the linear equation $a + bx = c + dx$ can be simplified by moving all terms involving the variable x to one side, e.g.,

$$a+bx=c+dx \quad \leftrightarrow \quad bx-dx=c-a \quad \leftrightarrow \quad x(b-d)=c-a \quad \leftrightarrow \quad x=\frac{c-a}{b-d}$$

where we first subtract dx and a from both sides, then factor out the variable x on the left-hand side, and finally divide left and right by $(b - d)$ to find the solution of x.

Thus, the general procedure for solving equations is to first bring one of the variables to one side of the equation, by modifying both sides equally. Finally one factors out the variable singled out on one side, and divides left and right by this factor.

This procedure is also valid for non-linear equations. Reconsider the general expression $f(x, y) = g(x, y)$, and note that one can always add something to (or subtract from) the left- and right-hand side simultaneously, e.g.,

$$f(x,y) = g(x,y) \quad \leftrightarrow \quad a + y + f(x,y) = a + y + g(x,y) \ ,$$

where we have added a constant and a variable to both sides. Likewise one can multiply (or divide) the left and right sides with (by) the same factor, e.g.,

$$f(x,y) = g(x,y) \quad \leftrightarrow \quad (a+y)f(x,y) = (a+y)g(x,y) \ .$$

1.1 Systems of linear equations

Systems of linear equations are solved similarly, by first solving one variable, and then substituting the solution in the other equations. For instance,

$$\begin{aligned} ax + by &= c \ , \\ dy &= ex \ , \end{aligned}$$

can be solved by starting with the simplest equation, $dy = ex$, giving $y = ex/d$. Substituting this solution into the first equation gives

$$ax + bex/d = c \quad \leftrightarrow \quad x(a + be/d) = c \quad \leftrightarrow \quad x = \frac{c}{a + be/d} \quad \leftrightarrow \quad x = \frac{cd}{ad + be} \ ,$$

where we factor out x on the left-hand side, divide by this factor, and simplify by multiplying numerator and denominator with d. Since $y = ex/d$ we find by substitution

$$y = \frac{e}{d}\,\frac{cd}{ad + be} = \frac{ec}{ad + be} \ ,$$

which completes the full solution of the system. For solving complicated systems of equations it is really important to start with the simplest equations: work from simple to complex rather than from top to bottom (see also below).

A common procedure is multiplying one equation with a clever factor, and then subtracting the equations from each other to obtain a new simple equation with just one variable. For instance,

$$\begin{aligned} ax + by &= c \ , \\ dx + ey &= f \ , \end{aligned}$$

can be solved by multiplying the second equation with a/d,

$$\frac{a}{d} \times (dx + ey) = \frac{a}{d} \times f \quad \leftrightarrow \quad ax + aey/d = af/d \ ,$$

which by design has the same term for x as the first equation. When we now subtract the left-hand side from this from the left-hand side of the first equation, and the two right-hand sides from each other, one obtains

$$by - aey/d = c - af/d \quad \leftrightarrow \quad bdy - aey = cd - af \quad \leftrightarrow \quad y(bd - ae) = cd - af \quad \leftrightarrow \quad y = \frac{cd - af}{bd - ae} \ ,$$

where we first mutiply both sides with d, then factor out y on the left-hand side, and finally divide both sides by $(bd - ae)$. Similarly, multiplying the first equation with e/b gives

$$\frac{e}{b} \times (ax + by) = \frac{e}{b} \times c \quad \leftrightarrow \quad aex/b + ey = ec/b \ ,$$

and subtracting this from the second equation gives

$$dx - aex/b = f - ec/b \quad \leftrightarrow \quad bdx - aex = fb - ec \quad \leftrightarrow \quad x(bd - ae) = fb - ec \quad \leftrightarrow \quad x = \frac{fb - ec}{bd - ae} \ ,$$

which by similar algebra completes the solution of the whole system.

As an alternative for the second step, one could also have substituted the solution of y, i.e., $\bar{y} = \frac{cd-af}{bd-ae}$, into the first equation to find the solution of x, i.e.,

$$ax + b\bar{y} = c \quad \leftrightarrow \quad x = \frac{c}{a} - \frac{b}{a}\,\bar{y} \quad \leftrightarrow \quad x = \frac{c}{a} - \frac{b}{a}\,\frac{cd - af}{bd - ae} \quad \leftrightarrow$$

$$x = \frac{c(bd - ae)}{a(bd - ae)} - \frac{b(cd - af)}{a(bd - ae)} \quad \leftrightarrow \quad x = \frac{-cae}{a(bd - ae)} + \frac{baf}{a(bd - ae)} \quad \leftrightarrow \quad x = \frac{fb - ec}{bd - ea} \ ,$$

where we first write x in terms of the above solution $\bar{y}$, by subtracting $b\bar{y}$ and dividing by a, then we fill in the solution for $\bar{y}$, give the first fraction, c/a, the same denominator as the second one (by multiplying numerator and denominator with $(bd - ae)$), add them, and simplify. This procedure involves more work, so one better thinks beforehand how to best solve a system of equations.

Note that subtracting one equality from another equality is nothing more than the application of our first rule: one can always add the same factor to both sides of an equation. Here the factor added (or subtracted) looks different on both sides, but its equality symbol tells us the two sides are just the same. Finally, solving systems of more than two equations just works the same way: work from the simplest to the most complex equation, solving variable by variable.

Exercises

1.1. Solve:
a. $ax + by = c$ for x or y.
b. $ax = b \ , \ cx + dy = e \ .$
c. $ax + by = cz \ , \ dx + ey = f \ , \ gz = hx \ .$

Check your answers using WolframAlpha. For instance '`solve a x + b y = c and d y = e x for x and y`' gives the solutions for one of the examples discussed above (note that Mathematica also considers cases where some of the parameters are zero). Actually, WolframAlpha is clever and will also understand what you mean when your type '`solve a x + b y = c, d y = e x`'.

2 Solving quadratic equations

An equation is called linear if the highest order of its variables is one, and is called quadratic if its highest order involves squared variables like ax^2 or by^2. Solving quadratic equations uses the same rules, for instance for solving $ax^2 - b = 0$, we obtain from

$$ax^2 - b = 0 \quad \leftrightarrow \quad ax^2 = b \quad \leftrightarrow \quad x^2 = b/a \quad \text{that} \quad x_{1,2} = \pm\sqrt{b/a},$$

by first adding b to both sides, then dividing both sides by a, and finally taking the square root of the left- and right-hand side.

For solving quadratic equations it is good to remember the famous expressions

$$(a + b)^2 = a^2 + 2ab + b^2 \ , \quad (a - b)^2 = a^2 - 2ab + b^2 \quad \text{and} \quad a^2 - b^2 = (a + b)(a - b) \ ,$$

which can easily be checked by multiplying the terms between the brackets and simplification. Many equations are rearranged by using these expressions.

For instance, the famous quadratic formula, $x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, for the solution of a quadratic equation can be obtained by working towards the first of these 3 expressions (see Wikipedia). Starting with the general quadratic equation,

$$ax^2 + bx + c = 0 \ ,$$

one multiplies the left- and right-hand side with $4a$,

$$4a^2x^2 + 4abx + 4ac = 0 \ ,$$

adds b^2 to both sides, and subtracts $4ac$ from both sides, to create the square $(2ab + b)^2$ on the left-hand side

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac \quad \leftrightarrow \quad (2ax + b)^2 = b^2 - 4ac \ .$$

The right-hand side is called the discriminant, i.e., when $b^2 - 4ac < 0$ there will be no real solution. If the discriminant is positive, one takes the square root left and right

$$2ax + b = \pm\sqrt{b^2 - 4ac} \quad \text{giving} \quad x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \ .$$

Note that this quadratic equation defines the points where the parabola, $y = ax^2 + bx + c$, intersects the horizontal x-axis. A parabola can indeed intersect twice, touch the x-axis once, or fail to intersect it (when $b^2 - 4ac < 0$).

2.1 Solving systems of quadratic equations

Systems of quadratic equations are solved in the same way as systems of linear equations: Manipulate both sides of the equations by the two rules defined above, and first solve the most simple equations.

For instance, solving x and y from

$$\begin{aligned} ax^2 - by &= c \\ dy^2 &= e \ , \end{aligned}$$

one could first solve $\bar{y} = \pm\sqrt{e/d}$ from the second equation, which is a positive or negative constant (not involving x), and then substitute this into the first equation

$$ax^2 - b\bar{y} = c \quad \leftrightarrow \quad ax^2 = b\bar{y} + c \quad \leftrightarrow \quad x^2 = b\bar{y}/a + c/a \quad \leftrightarrow \quad x = \pm\sqrt{b\bar{y}/a + c/a} \ ,$$

where we bring all terms not involving x to the right-hand side, divide left and right by a, take a square root, and finally substitute $\bar{y}$ to find four solutions:

$$x = \pm\sqrt{\frac{b}{a}\sqrt{\frac{e}{d}} + \frac{c}{a}} \quad \text{or} \quad x = \pm\sqrt{\frac{c}{a} - \frac{b}{a}\sqrt{\frac{e}{d}}} \ .$$

Note that substituting the pair of constants, $\bar{y}$, only at the end saves us a lot of writing.

The approach of multiplying an equation with a clever factor such that one can eliminate a variable by subtracting the modified equation from another equation works equally well with quadratic equations: For instance, solving x and y from

$$\begin{aligned} ax^2 - by &= c \\ dx^2 + ey &= f \ , \end{aligned}$$

where x and y have the same order in both equations, one could multiply the second equation with a/d to obtain

$$\frac{a}{d} \times (dx^2 + ey) = \frac{a}{d} \times f \quad \leftrightarrow \quad ax^2 + aey/d = af/d \ ,$$

and subtracting this from the first equation delivers

$$-by - aey/d = c - af/d \quad \leftrightarrow \quad bdy + aey = af - cd \quad \leftrightarrow \quad y(bd + ae) = af - cd \quad \leftrightarrow \quad y = \frac{af - cd}{ae + bd} \ ,$$

where we multiply left and right with $-d$, factor out y, and divide left and right by $(bd + ae)$.

Similarly, we could multiply the second equation with b/e to obtain

$$\frac{b}{e} \times (dx^2 + ey) = \frac{b}{e} \times f \quad \leftrightarrow \quad bdx^2/e + by = bf/e \ ,$$

and add that to the first equation

$$ax^2 + bdx^2/e = c + bf/e \quad \leftrightarrow \quad aex^2 + bdx^2 = ce + bf \quad \leftrightarrow \quad x^2(ae + bd) = ce + bf \quad \leftrightarrow \quad x^2 = \frac{ce + bf}{ae + bd} \ ,$$

where we multiply left and right with e, factor out x^2, and divide left and right by $(ae + bd)$. Finally taking the square root we obtain $x = \pm\sqrt{\frac{ce+bf}{ae+bd}} = \pm\frac{\sqrt{ce+bf}}{\sqrt{ae+bd}}$.

Exercises

2.1. Expand the following:
a. $(ax^2 + by)^2$
b. $(ax + b)^2 + (x - c)^2$
c. $(x + b)(x - b)$

2.2. Solve for x:
a. $(x - a)(x - b) = 0$
b. $-ax^2 - bx + c = 0$
c. $ax(b + x) = c$

3 Solving equations with fractions

Fractions are composed of a numerator and a denominator, and fractions with a complicated numerator can be split by separating its terms, e.g.,

$$\frac{a+b}{c+d} = \frac{a}{c+d} + \frac{b}{c+d} \ .$$

Different fractions can be added by giving them the same denominator,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + cb}{bd} \ ,$$

multiplication corresponds to multiplying the numerators and the denominators,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \ ,$$

and division is just the inverse of multiplication,

$$\frac{a}{b} : \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc} \ ,$$

where one should take care to not divide by zero, i.e., $bc \neq 0$.

Equations involving fractions are treated with the same rules as used above, i.e., treat both sides equally and bring one of the variables to one side, with one very natural addition, namely that **(3)** one can multiply (or divide) the numerator and denominator with the same factor. For example

$$\frac{ax}{b+x} = \frac{cax}{c(b+x)} \quad \text{or} \quad \frac{ax}{b+x} = \frac{a}{b/x + 1} \ ,$$

where we have multiplied numerator and denominator with c, or divided numerator and denominator by x. This can easily be checked, because the ratio of the numerator and denominator does not change when both are multiplied (or divided) by the same factor. (This is actually the same as multiplying both sides of an expression with the same factor). There is an obvious, but important, difference with expressions because one cannot add the same factor to the numerator and denominator, e.g.,

$$\frac{ax}{b+x} \neq \frac{ax+c}{b+c+x} \quad \text{and} \quad \frac{ax}{b+x} \neq \frac{ax-x}{b} \ ,$$

where we add c or $-x$ to the numerator and denominator. Thus to solve an expression like, $\frac{a}{x-1} = b$, one cannot add one to the denominator, but one can eliminate the fraction by multiplying left and right with $(x-1)$,

$$\frac{a}{x-1} = b \quad \leftrightarrow \quad a = b(x-1) \quad \leftrightarrow \quad \frac{a}{b} = x-1 \quad \leftrightarrow \quad 1 + \frac{a}{b} = x \ ,$$

and then divide left and right by b, and finally add one to both sides.

Solving equations with fractions just adds this one new rule to our procedure of treating left- and right-hand sides equally. For instance,

$$\frac{ax^2}{bx+cx^3} = d \quad \leftrightarrow \quad \frac{ax}{b+cx^2} = d \quad \leftrightarrow \quad ax = d(b+cx^2) \quad \leftrightarrow \quad 0 = cdx^2 - ax + bd \ ,$$

where we first divide numerator and denominator by x, multiply left and right side by $(b+cx^2)$, and subtract ax from both sides. The latter we can solve with the quadratic equation, i.e.,

$$x_{1,2} = \frac{a \pm \sqrt{a^2 - 4bcd^2}}{2cd} \ .$$

For a more complicated example consider solving

$$\frac{ax}{b+x} = \frac{dy}{e} \ ,$$

for x or for y. Solving for y gives

$$\frac{ax}{b+x} = \frac{dy}{e} \quad \leftrightarrow \quad y = \frac{aex}{d(b+x)} \ ,$$

when we divide left and right by d, and multiply left and right by e.

Solving for x is more cumbersome. First simplify, by multiplying left and right with e, then multiply left and right with $(b+x)$, i.e.,

$$\frac{ax}{b+x} = \frac{dy}{e} \quad \leftrightarrow \quad \frac{aex}{b+x} = dy \quad \leftrightarrow \quad aex = bdy + dxy \ ,$$

Next, bring all terms containing x to the left-hand side by subtracting dxy, and factor out x,

$$aex = bdy + dxy \quad \leftrightarrow \quad aex - dxy = bdy \quad \leftrightarrow \quad x(ae-dy) = bdy \ ,$$

to obtain that $x = \frac{bdy}{ae-dy}$ by dividing left and right with $(ae-dy)$. Because solving for x is so much more inconvenient that solving for y, this again illustrates that one has to think beforehand which variable is most easily solved from an expression.

Exercises

3.1. Working with fractions:
a. Expand into several terms $\frac{x+a}{x-a}$
b. Write as one term $\frac{a}{x-b} - \frac{a}{x+b}$
c. Write as one term $\frac{a}{x-b}\frac{a}{x+b}$

3.2. Mixed exercises:
a. Solve for x: $\frac{x+a}{x-a} = 0$
b. Simplify: $\frac{a}{x-b} - \frac{a}{x+b}$
c. Simplify: $\frac{a^2-b^2}{a+b}$

4 Equations involving powers, radicals, and logarithms

You will probably remember the following rules for raising a constant or variable to some number

$$x^a x^b = x^{a+b} \ , \quad \frac{x^a}{x^b} = x^{a-b} \ , \quad (x^a)^b = x^{ab} \ , \quad (xy)^a = x^a y^a \ ,$$

$$(x/y)^a = \frac{x^a}{y^a} \quad \text{and hence} \quad \sqrt{x/y} = \frac{\sqrt{x}}{\sqrt{y}} \ .$$

Radicals can also be written in a power notation, e.g., $\sqrt{a} = a^{1/2}$ and $\sqrt[3]{a} = a^{1/3}$, which can be convenient because the same rules as above can again be applied, e.g., $x\sqrt{x} = x^{3/2}$. Note that square roots of fractions can be simplified by using the third rule of multiplying numerator and denominator with the same factor, e.g.,

$$\sqrt{\frac{a}{b}} = \sqrt{\frac{ab}{b^2}} = \frac{1}{b}\ \sqrt{ab}$$

and that similarly square roots can be eliminated from simple denominators,

$$\frac{a\sqrt{b}}{\sqrt{c}} = \frac{a\sqrt{b}\sqrt{c}}{\sqrt{c}\sqrt{c}} = \frac{a\sqrt{bc}}{c} \ .$$

The inverse of raising something to a power is taking the logarithm, e.g., if $x^a = b$ we define $\log_x[b] = a$, where x is the 'base' of the logarithm. You are probably familiar with $\log_{10}[1000] = 3$ because $10^3 = 1000$. Similarly, $\log_2[8] = 3$ because $2^3 = 8$. A very elegant property of logarithms is that multiplications turn into additions (and hence divisions into subtractions). To perform complicated calculations before we had computers or pocket calculators, people used slide rules ('rekenlinealen' in Dutch) that where based upon this property (see Wikipedia). Indeed observe that if $x = 10^a$ and $y = 10^b$ that $xy = 10^{a+b}$. Taking the logarithm we see that $\log_{10}[x] = a$, $\log_{10}[y] = b$ and that $\log_{10}[xy] = a + b$, leading to the general formula

$$\log[xy] = \log[x] + \log[y] \quad \text{and} \quad \log[x/y] = \log[x] - \log[y] \ .$$

From the formula on the left one can also see that $\log[x^a] = a \log[x]$

Because $x^0 = 1$ we observe that $\log[1] = 0$, which means that

$$\log[1/a] = \log[1] - \log[a] = -\log[a] \ .$$

One can change base of a logarithm because $\log_b[x] = \log_k[x]/\log_k[b]$, where k is an arbitrary base. For example, to go from base 10 to base 2, one writes $\log_2[1000] = \log_{10}[1000]/\log_{10}[2] = \frac{3}{0.301} \simeq 10$, which is correct because $2^{10} = 1024 \simeq 10^3$.

There is an important base defining the natural logarithm, $\log_{\mathrm{e}}[x] = \ln[x]$, which is the inverse of the exponential function e^a, i.e., $\ln[\mathrm{e}^a] = a$. Exponential functions are ubiquitous in mathematics and biology (because the derivative of e^x is e^x; see below), and hence we frequently work with natural logarithms. Because $\ln[x]$ is even more natural than $\log_{10}[x]$ most programming languages (including R) use `log(x)` for $\ln[x]$, and `log10(x)` for $\log_{10}[x]$. The correction factor to go from $\log_{10}$ to ln is $1/\log_{10}[\mathrm{e}] \simeq 2.3$, i.e., $\ln[x] = 2.3 \log_{10}[x]$ (which is be important when exponential growth or decay is plotted on a log10 scale). Finally, note that $\mathrm{e}^{\ln[a]} = a$, and check that $10^{\log_{10}[1000]} = 1000$.

Exercises

4.1. Mixed exercises:

a. Solve for x: $a = b\mathrm{e}^{cx}$
b. Solve for x: $1 - \mathrm{e}^{-ax} = 1/2$ for $x > 0$ and $a > 0$.
c. Simplify $ax^b \times (x^c)^d$.

The second exercise can be checked with WolframAlpha by typing `solve 1-exp(-a x)=1/2 with x>0 and a>0`